The cohomology structure of an algebra entwined with a coalgebra

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Abstract

Two cochain complexes are constructed for an algebra A and a coalgebra C entwined with each other via the map $\psi : C \otimes A \to A \otimes C$. One complex is associated to an A-bimodule, the other to a C-bicomodule. In the former case the resulting complex can be considered as a ψ-twisted Hochschild complex of A, while for the latter one obtains a ψ-twist of the Cartier complex of C. The notion of a *weak comp algebra* is introduced by weakening the axioms of the Gerstenhaber comp algebra. It is shown that such a weak comp algebra is a cochain complex with two cup products that descend to the cohomology. It is also shown that the complexes associated to an entwining structure and A or C are examples of a weak comp algebra. Finally both complexes are combined in a double complex whose role in the deformation theory of entwining structures is outlined.

1 Introduction

An entwining structure [4] comprises of an algebra, a coalgebra and a map that entwines one with the other and satisfies some simple axioms. In many respects an entwining structure resembles a bialgebra or a comodule algebra of a bialgebra. Indeed, to any comodule algebra of a bialgebra, and hence to a bialgebra itself, there is associated a generic entwining structure, canonical in a certain sense. The aim of this paper is to reveal that entwining structures admit a rich cohomology theory, which, depending on the choice of ingredients in the entwining structure, can be viewed as the Hochschild

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cohomology of an algebra [10] or the Cartier cohomology of a coalgebra [5], and is reminiscent of the Gerstenhaber-Schack theory for bialgebras [9].

Recall from [4] that an *entwining structure* over a field k consists of an algebra A, a coalgebra C and a map $\psi : C \otimes A \to A \otimes C$ such that the following *bow-tie diagram* commutes.

$$
\begin{tikzcd}
 & C\otimes A\otimes A \arrow[ddl, "\psi\otimes A"'] \arrow[drr, "C\otimes\mu"] & & & & C\otimes C\otimes A \arrow[ddr, "C\otimes\psi"] & \\
 & & & C\otimes A \arrow[urr, "\Delta\otimes A"] \arrow[dd, "\psi"] \arrow[dr, "\epsilon\otimes A"] & & & \\
A\otimes C\otimes A \arrow[ddr, "A\otimes\psi"] & & C \arrow[ur, "C\otimes 1"] \arrow[dr, "1\otimes C"'] & & A & & C\otimes A\otimes C \arrow[ddl, "\psi\otimes C"] \\
 & & & A\otimes C \arrow[ur, "A\otimes\epsilon"'] \arrow[drr, "A\otimes\Delta"'] & & & \\
 & A\otimes A\otimes C \arrow[urr, "\mu\otimes C"'] & & & & A\otimes C\otimes C &
\end{tikzcd}
$$

Here and below we use the following notation. The product in A is denoted by μ, while the unit (both as an element of A and the map $k \to A$) is denoted by 1. For a coalgebra C, Δ is the coproduct, while ϵ is the counit. We use the Sweedler notation for action of Δ on elements of C, $\Delta(c) = c_{(1)} \otimes c_{(2)}$ (summation understood). Finally for any vector space V, the identity map $V \to V$ is denoted by V; we also implicitly identify $k \otimes V$ and $V \otimes k$ with V, use $\otimes$ for $\otimes_k$ and write V^n for $V^{\otimes n}$.

An entwining structure is denoted by $(A, C)_\psi$. To describe the action of ψ we use the following α-*notation*: $\psi(c \otimes a) = a_\alpha \otimes c^\alpha$ (summation over a Greek index understood), for all $a \in A$, $c \in C$, which proves very useful in concrete computations involving ψ. Reader is advised to check that the bow-tie diagram is equivalent to the following four explicit relations:

$$
\begin{aligned}
&\text{left pentagon: } (aa')_\alpha \otimes c^\alpha = a_\alpha a'_\beta \otimes c^{\alpha\beta}, \quad \text{left triangle: } 1_\alpha \otimes c^\alpha = 1 \otimes c,\\
&\text{right pentagon: } a_\alpha \otimes {c^\alpha}_{(1)} \otimes {c^\alpha}_{(2)} = a_{\beta\alpha} \otimes {c_{(1)}}^\alpha \otimes {c_{(2)}}^\beta, \quad \text{right triangle: } a_\alpha \epsilon(c^\alpha) = a\epsilon(c),
\end{aligned}
$$

for all $a, a' \in A$, $c \in C$.

One may (or perhaps even should) think of an entwining map ψ as a twist in the convolution algebra $\mathrm{Hom}(C, A)$. Namely, given an entwining structure, one can define the map $*_\psi : \mathrm{Hom}(C, A) \otimes \mathrm{Hom}(C, A) \to \mathrm{Hom}(C, A)$ via $(f *_\psi g)(c) = f(c_{(2)})_\alpha g({c_{(1)}}^\alpha)$, for all $f, g \in \mathrm{Hom}(C, A)$ and $c \in C$. One can easily check that $(\mathrm{Hom}(C, A), *_\psi)$ is an associative algebra with unit $1 \circ \epsilon$.

There are many examples of entwining structures. As a generic example one can refer to the following situation. Suppose C is a bialgebra and A is a right C-comodule algebra with the coaction ρ^A. Then $\psi : c \otimes a \mapsto a_{(0)} \otimes ca_{(1)}$, where $\rho^A(a) = a_{(0)} \otimes a_{(1)}$, entwines C with A. A special case of this situation is when A is an algebra and a coalgebra at the same time. Then A is a bialgebra if and only if $\psi : A \otimes A \to A \otimes A$, $\psi : a \otimes a' \mapsto a'_{(1)} \otimes aa'_{(2)}$ entwines A with itself. Furthermore, any algebra and a coalgebra can be provided with an entwining structure with ψ being the usual flip of tensor factors (for obvious reasons this can be called a *trivial* entwining structure). Interesting examples come from the generalisation of Hopf-Galois theory, motivated by the geometry of quantum (group) homogeneous spaces.

Example 1.1 [3] Let C be a coalgebra, A an algebra and a right C-comodule with the coaction $\rho^A : A \to A \otimes C$. Let $B := \{b \in A \mid \rho^A(ba) = b\rho^A(a)\}$ and assume that the canonical left A-module, right C-comodule map $\mathrm{can} : \ A \otimes_B A \to A \otimes C$, $a \otimes a' \mapsto a\rho^A(a')$, is bijective. Let $\psi : C \otimes A \to A \otimes C$ be a k-linear map given by $\psi(c \otimes a) = \mathrm{can}(\mathrm{can}^{-1}(1 \otimes c)a)$. Then $(A, C)_\psi$ is an entwining structure. The extension $B \hookrightarrow A$ is called a *coalgebra-Galois extension* (or a *C-Galois extension*) and is denoted by $A(B)^C$. This is a generalisation of the notion of a Hopf-Galois extension introduced in [12]. $(A, C)_\psi$ is called a *canonical entwining structure* associated to $A(B)^C$.

If C is a Hopf algebra and $A(B)^C$ is a Hopf-Galois extension, then the canonical entwining structure is the generic one described above. Also, any Hopf algebra is a Hopf-Galois extension of k, and the canonical entwining structure in this case is the one described above for a bialgebra. Dually we have

Example 1.2 [3] Let A be an algebra, C a coalgebra and a right A-module with the action $\rho_C : A \otimes C \to C$. Let $B := C/I$, where I is a coideal in C,

$$I := \mathrm{span}\{(c \cdot a)_{(1)}\xi((c \cdot a)_{(2)}) - c_{(1)}\xi(c_{(2)} \cdot a) \mid a \in A,\ c \in C,\ \xi \in C^*\},$$

and assume that the canonical left C-comodule, right A-module map $\mathrm{cocan} : \ C \otimes A \to C \Box_B C$, $c \otimes a \mapsto c_{(1)} \otimes c_{(2)} \cdot a$, is bijective. Let $\psi : C \otimes A \to A \otimes C$ be a k-linear map given by

$$\psi = (\epsilon \otimes A \otimes C) \circ (\mathrm{cocan}^{-1} \otimes C) \circ (C \otimes \Delta) \circ \mathrm{cocan}.$$

Then $(A, C)_\psi$ is an entwining structure. The coextension $C \twoheadrightarrow B$ is called an *algebra-Galois coextension* (or an *A-Galois coextension*) and is denoted by $C(B)_A$. $(A, C)_\psi$ is called a *canonical entwining structure* associated to $C(B)_A$.

In dealing with cohomology we will need A-bimodule (C-bicomodule resp.) structures on $A\otimes C^n$ ($C\otimes A^n$ resp.). These are defined as follows. Given an entwining structure $(A,C)_\psi$ define two infinite families of maps

$$\psi^n = (A^{n-1}\otimes\psi)\circ(A^{n-2}\otimes\psi\otimes A)\circ\cdots\circ(\psi\otimes A^{n-1}) : C\otimes A^n\to A^n\otimes C, \quad n\geq 1,$$

$$\psi_n = (\psi\otimes C^{n-1})\circ(C\otimes\psi\otimes C^{n-2})\circ\cdots\circ(C^{n-1}\otimes\psi) : C^n\otimes A\to A\otimes C^n, \quad n\geq 1.$$

The axioms of an entwining structure imply that for all $n>0$, $A\otimes C^n$ is an A-bimodule with the left action $\rho_n^L = \mu\otimes C^n$ and the right action $\rho_n^R = (\mu\otimes C^n)\circ(A\otimes\psi_n)$. Furthermore, $C\otimes A^n$ is a C-bicomodule with the left coaction $\rho^n_L = \Delta\otimes A^n$ and the right coaction $\rho^n_R = (C\otimes\psi_n)\circ(\Delta\otimes A^n)$. We will always consider $A\otimes C^n$ ($C\otimes A^n$ resp.) as bimodules (bicomodules resp.) with the above structures. Also, for any vector space V the space $A\otimes V\otimes A$ ($C\otimes V\otimes C$ resp.) will be considered as an A-bimodule (C-bicomodule resp.) with the obvoius structure maps $\mu\otimes V\otimes A$ and $A\otimes V\otimes\mu$ ($\Delta\otimes V\otimes C$, $C\otimes V\otimes\Delta$ resp.).

Yet another consequence of the axioms of an entwining structure is the following

Lemma 1.3 *For all $n\geq 1$, $0\leq j\leq n-1$ the following two diagrams*

$$\begin{tikzcd}
C\otimes A^{n+1} \arrow[r, "C\otimes A^j\otimes\mu\otimes A^{n-j-1}"] \arrow[d, "\rho_R^{n+1}"'] & C\otimes A^n \arrow[d, "\rho_R^n"] \\
C\otimes A^{n+1}\otimes \arrow[r, "C\otimes A^j\otimes\mu\otimes A^{n-j-1}\otimes C"] & C\otimes A^n\otimes C
\end{tikzcd}$$

$$\begin{tikzcd}
A\otimes C^n\otimes A \arrow[r, "A\otimes C^j\otimes\Delta\otimes C^{n-j-1}\otimes A"] \arrow[d, "\rho_n^R"'] & A\otimes C^{n+1}\otimes A \arrow[d, "\rho_{n+1}^R"] \\
A\otimes C^n \arrow[r, "A\otimes C^j\otimes\Delta\otimes C^{n-j-1}"] & A\otimes C^{n+1}
\end{tikzcd}$$

commute.

The paper is organised as follows. In the next section we construct the cochain complex $C_\psi(A,M)$ associated to an entwining structure $(A,C)_\psi$ and an A-bimodule M. We study its relation to the Hochschild complex of A as well as analyse its structure in the case of the canonical entwining structure associated to a C-Galois extension $A(B)^C$. In particular we show that if $B=k$ this complex provides a resolution of M. In

Section 3 we dualise the construction of Section 2 and describe a complex $A_\psi(C,V)$ associated to $(A,C)_\psi$ and a C-bicomodule V. Section 4 is devoted to studies of the complex $C_\psi(A) = C_\psi(A,A)$. We define two cup products in $C_\psi(A)$ which descend to the cohomology. In particular we show that in the cohomology one product is a graded twist of the other. All this is done with the help of the notion of a *weak comp algebra*, which generalises the notion of a right comp algebra [8] or a pre-Lie system [7] (see recent review [11] of comp algebras). In Section 5 we define an *equivariant* complex as a subcomplex of $C_\psi(A)$ on which both cup products coincide, so that the corresponding algebra in such an *equivariant* cohomolgy is graded commutative. This extends the classic result of Gerstenhaber [7]. Finally in Section 6 we define a double complex and outline its role in the deformation theory of entwining structures.

2 Module valued cohomology of an entwining structure

In this section we associate a cochain complex to an entwining structure $(A,C)_\psi$ and an A-bimodule M.

Recall that the bar resolution of an algebra A is a chain complex $\mathrm{Bar}(A) = (\mathrm{Bar}_\bullet(A), \delta)$, where

$$\mathrm{Bar}_n(A) = A^{n+2}, \quad \delta_n = \sum_{k=0}^{n} (-1)^k A^k \otimes \mu \otimes A^{n-k} : A^{n+2} \to A^{n+1}.$$

Since δ is an A-bimodule map, one can define a chain complex $\mathrm{Bar}^\psi(A) = (\mathrm{Bar}^\psi_\bullet(A), \delta)$ via $\mathrm{Bar}^\psi(A) = (A \otimes C) \otimes_A \mathrm{Bar}(A)$. Here $A \otimes C$ is viewed as an A-bimodule as explained in the introduction. Explicitly, $\mathrm{Bar}^\psi_n(A) = A \otimes C \otimes A^{n+1}$ and

$$\delta_n = (\mu \otimes C \otimes A^n) \circ (A \otimes \psi \otimes A^n) + \sum_{k=1}^{n} (-1)^k (A \otimes C \otimes A^{k-1} \otimes \mu \otimes A^{n-k}).$$

Even more explicitly, using the α-notation, one can write

$$\begin{aligned} \delta_n(a^0 \otimes c \otimes a^1 \otimes \cdots \otimes a^{n+1}) &= a^0 a^1_\alpha \otimes c^\alpha \otimes a^2 \otimes \cdots \otimes a^{n+1} \\ &\quad + \sum_{i=1}^{n} (-1)^i a^0 \otimes c \otimes a^1 \otimes \cdots \otimes a^i a^{i+1} \otimes \cdots \otimes a^{n+1}. \end{aligned}$$

Lemma 2.1 $\mathrm{Bar}^\psi(A)$ *is a resolution of both* $A \otimes C$ *and* A. *Furthermore,* δ *is an* A-*bimodule map.*

Proof. $\mathrm{Bar}^\psi(A)$ is an acyclic complex, because there is a contracting homotopy $h_n : \mathrm{Bar}^\psi_n(A) \to \mathrm{Bar}^\psi_{n+1}(A)$, given as $h_n = (-1)^n A\otimes C\otimes A^{n+1}\otimes 1$. The left pentagon in the bow-tie diagram implies that the map $(\mu\otimes C)\circ(A\otimes\psi) : A\otimes C\otimes A\to A\otimes C$ is an augmentation. Similarly, the right triangle implies that $\mu\circ(A\otimes\epsilon\otimes A) : A\otimes C\otimes A\to A$ is an augmentation as well. It is obvious that all the δ_n are A-bimodule maps. □

Next we use the resolution $\mathrm{Bar}^\psi(A)$ to construct one of the main cochain complexes studied in this paper. Let M be an A-bimodule. Define the cochain complex $C_\psi(A,M) = (C_\psi(A,M)^\bullet, d)$ by $C_\psi(A,M) = {}_A\mathrm{Hom}_A(\mathrm{Bar}^\psi(A), M)$, where ${}_A\mathrm{Hom}_A$ denotes the Hom-bifunctor from the category of A-bimodules to the category of k-vector spaces. Using the natural identification ${}_A\mathrm{Hom}_A(A\otimes C\otimes A^{n+1}, M) = \mathrm{Hom}(C\otimes A^n, M)$, one explicitly obtains

$$C^n_\psi(A,M) = \mathrm{Hom}(C\otimes A^n, M),\quad d^n : \mathrm{Hom}(C\otimes A^n, M)\to \mathrm{Hom}(C\otimes A^{n+1}, M),$$

$$d^n f = {}_M\rho\circ(A\otimes f)\circ(\psi\otimes A^n) + \sum_{k=1}^n (-1)^k f\circ(C\otimes A^{k-1}\otimes\mu\otimes A^{n-k}) + (-1)^{n+1}\rho_M\circ(f\otimes A),$$

where ρ_M (${}_M\rho$ resp.) denotes the right (left) action of A on M. Even more explicitly, using the α-notation, one has

$$\begin{aligned} d^n f(c, a^1, \dots, a^{n+1}) &= a^1_\alpha\cdot f(c^\alpha, a^2,\dots,a^{n+1}) + \sum_{i=1}^n (-1)^i f(c, a^1,\dots, a^i a^{i+1},\dots,a^{n+1}) \\ &\quad +(-1)^{n+1} f(c, a^1,\dots,a^n)\cdot a^{n+1}. \end{aligned}$$

To save the space we write $f(\cdot,\cdot,\dots,\cdot)$ for $f(\cdot\otimes\cdot\otimes\cdots\otimes\cdot)$, etc. There is a close relationship between the complex $C_\psi(A,M)$ and the Hochschild complex of A. Firstly, for any A-bimodule M consider the A-bimodule $\mathrm{Hom}(C,M)$ with the structure maps:

$$(f\cdot a)(c) = f(c)\cdot a,\qquad (a\cdot f)(c) = a_\alpha\cdot f(c^\alpha).$$

It is an easy exercise in the α-notation to verify that $\mathrm{Hom}(C,M)$ is an A-bimodule indeed. Identify $\mathrm{Hom}(A^n, \mathrm{Hom}(C,M))$ with $\mathrm{Hom}(C\otimes A^n, M)$ by the natural isomorphism:

$$\theta(f)(c, a^1,\dots,a^n) = f(c)(a^1,\dots,a^n),\quad \theta^{-1}(g)(a^1,\dots,a^n)(c) = g(c, a^1,\dots,a^n).$$

Then $C_\psi(A,M)$ is the Hochschild complex of A with values in $\mathrm{Hom}(C,M)$.

Secondly, the Hochschild complex over A with values in M is included in the complex $C_\psi(A,M)$. More precisely one has

Lemma 2.2 *Let $C(A,M)$ be the Hochschild complex over A with values in M. Then the map $j : C(A,M) \to C_\psi(A,M)$ given by*

$$j^n : \mathrm{Hom}(A^n, M) \to \mathrm{Hom}(C \otimes A^n, M), \qquad j^n : f \mapsto \epsilon \otimes f,$$

is a monomorphism of differential complexes.

Proof. Clearly, j is injective. The fact that j is the map between cochain complexes follows from the right triangle in the bow-tie diagram. □

The cohomology of the complex $C_\psi(A,M)$ is denoted by $H_\psi(A,M)$ and is called an *entwined cohomology of A with values in M.*

Proposition 2.3 *For an entwining structure $(A,C)_\psi$, $A\otimes C$ is a projective A-bimodule if and only if $H^1_\psi(A,M) = 0$ for all A-bimodules M.*

Proposition 2.3 will follow from the following two lemmas.

Lemma 2.4 *For an entwining structure $(A,C)_\psi$ the following statements are equivalent:*

(1) $A \otimes C$ is a projective A-bimodule.

(2) The sequence of A-bimodule maps

$$0 \longrightarrow \ker \rho_1^R \longrightarrow A\otimes C\otimes A \xrightarrow{\rho_1^R} A\otimes C \longrightarrow 0$$

is split exact.

(3) There exists a 0-cocycle $\chi \in C^0_\psi(A, A\otimes A)$ such that $\mu\circ\chi = 1\circ\epsilon$.

Proof. The equivalence of the first two assertions is clear since k is the field so that $A\otimes C\otimes A$ is a free A-bimodule. Suppose that (2) holds, i.e., there is an A-bimodule map $\nu : A\otimes C \to A\otimes C\otimes A$ such that $\rho_1^R \circ \nu = A\otimes C$. Define $\bar\chi = \nu\circ(1\otimes C)$ and $\chi = (A\otimes\epsilon\otimes A)\circ\bar\chi$. Then for all $a\in A$, $c\in C$ one has:

$$a_\alpha \bar\chi(c^\alpha) = a_\alpha \nu(1\otimes c^\alpha) = \nu((1\otimes c)\cdot a) = \nu(1\otimes c)a = \bar\chi(c)a.$$

Applying $A\otimes\epsilon\otimes A$ to both sides of this equality one immediately obtains that χ is a 0-cocycle. The normalisation of χ follows from the equality $\rho_1^R\circ\nu = A\otimes C$ applied to $1\otimes c$ and the right triangle in the bow-tie diagram.

Now suppose that (3) holds. Denote $\chi(c) = c^{(\bar 1)}\otimes c^{(\bar 2)}$ (summation understood), and define $\nu : A\otimes C\to A\otimes C\otimes A$, $a\otimes c \mapsto a{c_{(2)}}^{(\bar 1)}{}_\alpha \otimes {c_{(1)}}^\alpha \otimes {c_{(2)}}^{(\bar 2)}$. Then for all $a\in A$, $c\in C$,

$$\rho_1^R\circ\nu(a\otimes c) = a{c_{(2)}}^{(\bar 1)}{}_\alpha {c_{(2)}}^{(\bar 2)}{}_\beta \otimes {c_{(1)}}^{\alpha\beta} = a({c_{(2)}}^{(\bar 1)}{c_{(2)}}^{(\bar 2)})_\alpha \otimes {c_{(1)}}^\alpha = a\otimes c,$$

where we used the left pentagon and the left triangle in the bow-tie diagram together with the normalisation of χ. Therefore ν splits ρ_1^R. Clearly, ν is a left A-module map. Furthermore for all $a, a' \in A$, $c \in C$ we have:

$$\begin{aligned}
\nu((a\otimes c)\cdot a') &= a a'_\beta {c^\beta}_{(2)}{}^{(\bar{1})}{}_\alpha \otimes {c^\beta}_{(1)}{}^\alpha \otimes {c^\beta}_{(2)}{}^{(\bar{2})} \\
&= a a'_{\beta\gamma} {c_{(2)}}^{\beta(\bar{1})}{}_\alpha \otimes {c_{(1)}}^{\gamma\alpha} \otimes {c_{(2)}}^{\beta(\bar{2})} \\
&= a(a'_\beta {c_{(2)}}^{\beta(\bar{1})})_\alpha \otimes {c_{(1)}}^\alpha \otimes {c_{(2)}}^{\beta(\bar{2})} \\
&= a {c_{(2)}}^{(\bar{1})}{}_\alpha \otimes {c_{(1)}}^\alpha \otimes {c_{(2)}}^{(\bar{2})} a' = \nu(c\otimes a)a',
\end{aligned}$$

where we used the right pentagon to derive the second equality, the left pentagon to derive the third one and finally the fact that χ is a 0-cocycle to obtain the fourth equality. This proves that ν is an A-bimodule splitting as required. □

Lemma 2.5 *For an entwining structure $(A, C)_\psi$ and an A-bimodule M let $B^n_\psi(A, M)$ denote the space of n-coboundaries and $Z^n_\psi(A, M)$ denote the space of n-cocycles in $C^n_\psi(A, M)$. Let $D_\psi : C \otimes A \to A \otimes C \otimes A$ be a linear map given by $D_\psi : c \otimes a \mapsto 1 \otimes c \otimes a - a_\alpha \otimes c^\alpha \otimes 1$. Then:*

(1) The map $\theta : {}_A\mathrm{Hom}_A(\ker \rho_1^R, M) \to Z^1_\psi(A, M)$, $f \mapsto f \circ D_\psi$ is a bijection.

(2) $\theta^{-1}(B^n_\psi(A, M)) = \{f\,|_{\ker \rho_1^R} :\ f \in {}_A\mathrm{Hom}_A(A \otimes C \otimes A, M)\}$.

Proof. Throughout the proof of this lemma, $x = \sum_i a^i \otimes c_i \otimes \tilde{a}^i$ is an arbitrary element of $\ker \rho_1^R$. Notice that $\sum_i a^i \tilde{a}^i_\alpha \otimes c_i^\alpha = 0$.

(1) One easily finds that for all $f \in {}_A\mathrm{Hom}_A(\ker \rho_1^R, M)$, $d\theta(f) = 0$ so that the map θ is well-defined. Consider the map $\bar{\theta} : Z^1_\psi(A, M) \to {}_A\mathrm{Hom}_A(\ker \rho_1^R, M)$, given by $\bar{\theta}(\chi)(x) = \sum_i a^i \cdot \chi(c_i, \tilde{a}^i)$. Clearly, $\bar{\theta}(\chi)$ is a left A-module map. Since χ is a 1-cocycle, we have for all $a \in A$,

$$\bar{\theta}(\chi)(xa) = \sum_i a^i \cdot \chi(c_i, \tilde{a}^i a) = \sum_i a^i \cdot \chi(c_i, \tilde{a}^i)a + \sum_i a^i \tilde{a}^i_\alpha \cdot \chi(c_i^\alpha, a) = \bar{\theta}(\chi)(x) \cdot a.$$

Therefore $\bar{\theta}$ is well-defined. For any $\chi \in Z^1_\psi(A, M)$ one easily finds that for all $c \in C$, $\chi(c, 1) = 0$. Using this fact one obtains

$$\theta \circ \bar{\theta}(\chi)(c, a) = \bar{\theta}(\chi)(1 \otimes c \otimes a - a_\alpha \otimes c^\alpha \otimes 1) = \chi(c, a) - a_\alpha \chi(c^\alpha, 1) = \chi(c, a),$$

as well as

$$\bar{\theta} \circ \theta(f)(x) = \sum_i a^i \theta(f)(c_i, \tilde{a}^i) = f(x - \sum_i a^i \tilde{a}^i_\alpha \otimes c^\alpha \otimes 1) = f(x).$$

Therefore $\bar{\theta}$ is the inverse of θ, $\bar{\theta} = \theta^{-1}$.

(2) Suppose $\chi = -df$ for some $f \in \mathrm{Hom}(C, M)$. Then $\theta^{-1}(\chi)(x) = -\sum_i a^i \cdot df(c_i, \tilde{a}^i) = \sum_i a^i \cdot f(c_i) \cdot \tilde{a}^i - \sum_i a^i \tilde{a}^i_\alpha \cdot f(c_i^\alpha) = \sum_i a^i \cdot f(c_i) \cdot \tilde{a}^i$. The result then follows from the isomorphism $\mathrm{Hom}(C, M) \cong {}_A\mathrm{Hom}_A(A \otimes A, \mathrm{Hom}(C, M)) \cong {}_A\mathrm{Hom}_A(A \otimes C \otimes A, M)$ given by $f \mapsto \ell_f$, where $\ell_f(a \otimes c \otimes a') = a \cdot f(c) \cdot a'$. □

Now, using Lemma 2.4 and Lemma 2.5, Proposition 2.3 can be proven by the same reasoning as Proposition 11.5 in [13]. Namely, if $H^1_\psi(A, \ker \rho_1^R) = 0$, then, by Lemma 2.5, A-bimodule endomorphisms of $\ker \rho_1^R$ equal $\{f \mid_{\ker \rho_1^R} : \ f \in {}_A\mathrm{Hom}_A(A \otimes C \otimes A, \ker \rho_1^R)$. This means that there exists $f \in {}_A\mathrm{Hom}_A(A \otimes C \otimes A, \ker \rho_1^R)\}$ such that $f \mid_{\ker \rho_1^R} = \ker \rho_1^R$, i.e., the sequence in Lemma 2.4(2) splits. Thus $A \otimes C$ is a projective A-bimodule, by Lemma 2.4. Conversely, if there is an extension $f \in {}_A\mathrm{Hom}_A(A \otimes C \otimes A, \ker \rho_1^R)$ of the identity mapping $\ker \rho_1^R$, then for any A-bimodule M, every $g \in {}_A\mathrm{Hom}_A(\ker \rho_1^R, M)$ has the form $h \mid_{\ker \rho_1^R}$, where $h = g \circ f \in {}_A\mathrm{Hom}_A(A \otimes C \otimes A, M)$. By Lemma 2.5 $B^1_\psi(A, M) = Z^1_\psi(A, M)$, for any A-bimodule M. This completes the proof of Proposition 2.3.

As an example we compute the (canonical) entwined cohomology of a C-Galois extension.

Proposition 2.6 *Let $(A, C)_\psi$ be the canonical entwining structure associated to a C-Galois extension $A(B)^C$ in Example 1.1, and let M be an A-bimodule. Then*

(1) $H^0_\psi(A, M) = M^B := \{m \in M \mid \forall b \in B,\ b \cdot m = m \cdot b\}$.

(2) If A is a C-Galois object, i.e., $B = k$, then $H^n_\psi(A, M) = 0$, for all $n > 0$.

Proof. (1) Denote the action of the *translation map* $\tau := \mathrm{can}^{-1} \circ (1 \otimes C) : C \to A \otimes_B A$ by $\tau(c) = c^{(1)} \otimes c^{(2)}$ (summation understood). Notice that for all $a \in A$, $c \in C$, $c^{(1)} {c^{(2)}}_{(0)} \otimes {c^{(2)}}_{(1)} = 1 \otimes c$, $a_{(0)} {a_{(1)}}^{(1)} \otimes {a_{(1)}}^{(2)} = 1 \otimes a$ (cf. [14, 3.4 Remark (2)(a)]), and $\psi(c \otimes a) = c^{(1)} (c^{(2)} a)_{(0)} \otimes (c^{(2)} a)_{(1)}$, where we use the Sweedler notation for the coaction, $\rho^A(a) = a_{(0)} \otimes a_{(1)}$ (summation understood). Consider the map

$$\theta : M^B \to H^0_\psi(A, M), \qquad m \mapsto c^{(1)} \cdot m \cdot c^{(2)}.$$

The map θ is well-defined because $m \in M^B$ and furthermore,

$$\begin{aligned} d\theta(m)(c \otimes a) &= a_\alpha \cdot \theta(m)(c^\alpha) - \theta(m)(c) \cdot a \\ &= c^{(1)} (c^{(2)} a)_{(0)} {(c^{(2)} a)_{(1)}}^{(1)} \cdot m \cdot {(c^{(2)} a)_{(1)}}^{(2)} - c^{(1)} \cdot m \cdot c^{(2)} a \\ &= c^{(1)} \cdot m \cdot c^{(2)} a - c^{(1)} \cdot m \cdot c^{(2)} a = 0, \end{aligned}$$

so that $\theta(m)$ is a zero-cocycle, hence belongs to the zero-cohomology group.

Let $\rho^A(1) = 1_{(0)} \otimes 1_{(1)}$. We claim that the map

$$\theta^{-1} : H^0_\psi(A, M) \to M^B, \qquad \theta^{-1}(f) = 1_{(0)} \cdot f(1_{(1)})$$

is the inverse of θ. Firstly we need to check that $\theta^{-1}(f)$ is in the centraliser of B in M. The key observation needed here is that A is an $(A, C)_\psi$-module, i.e., for all $a, a' \in A$, $\rho^A(aa') = a_{(0)}a'_\alpha \otimes a_{(1)}{}^\alpha$. In particular this implies that for all $a \in A$, $\rho^A(a) = 1_{(0)}a_\alpha \otimes 1_{(1)}{}^\alpha$, and for all $b \in B$, $b1_{(0)} \otimes 1_{(1)} = 1_{(0)}b_\alpha \otimes 1_{(1)}{}^\alpha$ (cf. [2]). Since f is a zero-cocycle we have for any $b \in B$:

$$0 = 1_{(0)} \cdot df(1_{(1)}, b) = 1_{(0)}b_\alpha \cdot f(1_{(1)}{}^\alpha) - 1_{(0)} \cdot f(1_{(1)}) \cdot b = b \cdot \theta^{-1}(f) - \theta^{-1}(f) \cdot b,$$

so that $\theta^{-1}(f) \in M^B$ as claimed. Furthermore, since $\rho^A(a) = 1_{(0)}a_\alpha \otimes 1_{(1)}{}^\alpha$, we have for all 0-cocycles f, $a_{(0)} \cdot f(a_{(1)}) = 1_{(0)} \cdot f(1_{(1)}) \cdot a$. In particular we have

$$f(c) = c^{(1)}c^{(2)}{}_{(0)} \cdot f(c^{(2)}{}_{(1)}) = c^{(1)}1_{(0)} \cdot f(1_{(1)}) \cdot c^{(2)}.$$

Therefore

$$\theta \circ \theta^{-1}(f)(c) = c^{(1)} \cdot \theta^{-1}(f) \cdot c^{(2)} = c^{(1)}1_{(0)} \cdot f(1_{(1)}) \cdot c^{(2)} = f(c),$$

$$\theta^{-1} \circ \theta(m) = 1_{(0)} \cdot \theta(m)(1_{(1)}) = 1_{(0)}1_{(1)}{}^{(1)} \cdot m \cdot 1_{(1)}{}^{(2)} = m \cdot 1 = m.$$

(2) For any $n > 1$ consider the map $h^n : C^n_\psi(A, M) \to C^{n-1}_\psi(A, M)$, given by

$$h^n(f)(c, a^1, \dots, a^{n-1}) = c^{(1)}1_{(0)} \cdot f(1_{(1)}, c^{(2)}, a^1, \dots, a^{n-1}).$$

We will show that h is a contracting homotopy, i.e., $h^{n+1}d^n + d^{n-1}h^n = C^n_\psi(A, M)$. We use properties of the translation map listed above and the definitions of d and h to compute

$$\begin{aligned}
d^{n-1}h^n(f)(c, a^1, \dots, a^n) \;\; &= \;\; c^{(1)}(c^{(2)}a^1)_{(0)} \cdot h^n(f)((c^{(2)}a^1)_{(1)}, a^2, \dots, a^n) \\
&+ \sum_{k=1}^{n-1}(-1)^k h^n(f)(c, a^1, \dots, a^k a^{k+1}, \dots, a^n) + (-1)^n h^n(f)(c, a^1, \dots, a^{n-1}) \cdot a^n \\
&= c^{(1)}(c^{(2)}a^1)_{(0)}(c^{(2)}a^1)_{(1)}{}^{(1)}1_{(0)} \cdot f(1_{(1)}, (c^{(2)}a^1)_{(1)}{}^{(2)}, a^2, \dots, a^n) \\
&+ \sum_{k=1}^{n-1}(-1)^k c^{(1)}1_{(0)} \cdot f(1_{(1)}, c^{(2)}, a^1, \dots, a^k a^{k+1}, \dots, a^n) \\
&+ (-1)^n c^{(1)}1_{(0)} \cdot f(1_{(1)}, c^{(2)}, a^1, \dots, a^{n-1}) \cdot a^n
\end{aligned}$$

$$= c^{(1)}1_{(0)} \cdot f(1_{(1)}, c^{(2)}a^1, a^2, \dots, a^n) + (-1)^n c^{(1)}1_{(0)} \cdot f(1_{(1)}, c^{(2)}, a^1, \dots, a^{n-1}) \cdot a^n$$
$$+ \sum_{k=1}^{n-1} (-1)^k c^{(1)}1_{(0)} \cdot f(1_{(1)}, c^{(2)}, a^1, \dots, a^k a^{k+1}, \dots, a^n)$$
$$= -c^{(1)}1_{(0)} \cdot d^n f(1_{(1)}, c^{(2)}, a^1, \dots, a^n) + c^{(1)}1_{(0)}{c^{(2)}}_{\alpha} \cdot f({1_{(1)}}^{\alpha}, a^1, \dots, a^n)$$
$$= -h^{n+1}(d^n f)(c, a^1, \dots, a^n) + c^{(1)}{c^{(2)}}_{(0)} \cdot f({c^{(2)}}_{(1)}, a^1, \dots, a^n)$$
$$= -h^{n+1}(d^n f)(c, a^1, \dots, a^n) + f(c, a^1, \dots, a^n).$$

Therefore $h^\bullet$ is a contracting homotopy, so that for all $n > 0$, $H^n_\psi(A, M) = 0$ as claimed. □

In particular, Proposition 2.6 implies that if A is a Hopf algebra and $\psi : A \otimes A \to A \otimes A$, $a \otimes a' \to {a'}_{(1)} \otimes a{a'}_{(2)}$, then $H^0_\psi(A, M) = M$ and $H^n_\psi(A, M) = 0$ for all $n > 0$. Furthermore, in view of Proposition 2.3, Proposition 2.6 implies that for a C-Galois object A, $A \otimes C$ is a projective A-bimodule. Notice also that Proposition 2.6(1) states that the entwined zero-cohomology group of $A(B)^C$ is the same as the Hochschild zero-cohomology group of B.

As another example we compute the zero cohomology group of an A-Galois coextension with values in A.

Example 2.7 *Suppose $(A, C)_\psi$ is the canonical entwining structure associated to an A-Galois coextension $C(B)_A$. Then $H^0_\psi(A, A) = {}^B\mathrm{End}_A(C)$ (the space of left B-comodule right A-module endomorphisms of C).*

Proof. In general $H^0_\psi(A, A) = \{\phi \in \mathrm{Hom}(C, A) \mid \forall c \in C, a \in A \quad a_\alpha \phi(c^\alpha) = \phi(c)a\}$. Now, [2, Theorem 2.4*] yields the assertion. □

3 Comodule valued cohomology of an entwining structure

The construction of the previous section can be dualised to produce a ψ-twisted cohomology of a coalgebra. Thus the aim of this section is to describe a cochain complex associated to an entwining structure $(A, C)_\psi$ and a C-bicomodule V.

Consider the cobar resolution of a coalgebra C, $\mathrm{Cob}(C) = (\mathrm{Cob}^\bullet(C), \bar{\delta})$, where

$$\mathrm{Cob}^n(C) = C^{n+2}, \quad \bar{\delta}^n = \sum_{k=0}^{n} (-1)^k C^k \otimes \Delta \otimes C^{n-k} : C^{n+2} \mapsto C^{n+3}.$$

One easily checks that $\bar{\delta}$ is a C-bicomodule map so that one can define the cochain complex $\mathrm{Cob}_\psi(C) = (\mathrm{Cob}^\bullet_\psi(C), \bar{\delta})$ via $\mathrm{Cob}_\psi(C) = (C \otimes A)\Box_C \mathrm{Cob}(C)$, where $\Box_C$ denotes the cotensor product over C. Explicitly, $\mathrm{Cob}^n_\psi(C) = C \otimes A \otimes C^{n+1}$ and

$$\bar{\delta}^n = (C \otimes \psi \otimes C^n) \circ (\Delta \otimes A \otimes C^n) + \sum_{k=1}^{n} (-1)^k C \otimes A \otimes C^{k-1} \otimes \Delta \otimes C^{n-k}.$$

As in the case of the ψ-twisted bar resolution $\mathrm{Bar}^\psi(A)$ one has,
Lemma 3.1 $\mathrm{Cob}_\psi(C)$ *is a resolution of* C *and* $C \otimes A$*. Furthermore,* $\bar{\delta}$ *is a* C*-bicomodule map.*

Proof. The contracting homotopy is $h^n : \mathrm{Cob}^n_\psi(C) \to \mathrm{Cob}^{n-1}_\psi(C)$, $h^n = (-1)^{n+1} C \otimes A \otimes C^n \otimes \epsilon$, while the augmentations are $(C \otimes 1 \otimes C) \circ \Delta$ and $(C \otimes \psi) \circ (\Delta \otimes A)$. □

Let V be a C-bicomodule. Define the cochain complex $A_\psi(C, V) = (A_\psi(C, V)^\bullet, \bar{d})$ by $A_\psi(C, V) = {}^C\mathrm{Hom}^C(V, \mathrm{Cob}_\psi(C))$, where ${}^C\mathrm{Hom}^C$ denotes the Hom-bifunctor from the category of C-bicomodules to the category of k-vector spaces. Using the natural identification ${}^C\mathrm{Hom}^C(C, C \otimes A \otimes C^{n+1}) = \mathrm{Hom}(V, A \otimes C^n)$, one explicitly obtains

$$A^n_\psi(C, V) = \mathrm{Hom}(V, A \otimes C^n), \quad \bar{d}^n : \mathrm{Hom}(V, A \otimes C^n) \to \mathrm{Hom}(V, A \otimes C^{n+1}),$$

$$\bar{d}^n f = (C \otimes f) \circ (\psi \otimes C^n) \circ {}^V\rho + \sum_{k=1}^{n} (-1)^k (A \otimes C^{k-1} \otimes \Delta \otimes C^{n-k}) \circ f + (-1)^{n+1} (f \otimes C) \circ \rho^V,$$

where ρ^V (${}^V\rho$ resp.) denotes the right (left) coaction of C on V.

The Cartier complex over C with values in V (cf. [5]) is included in the complex $A_\psi(C, V)$. More precisely one has

Lemma 3.2 *Let* $C(C, V)$ *be the Cartier complex over* C *with values in* V*. Then the map* $\bar{j} : C(C, V) \to A_\psi(C, V)$ *given by*

$$\bar{j}^n : \mathrm{Hom}(V, C^n) \to \mathrm{Hom}(V, A \otimes C^n), \qquad \bar{j}^n : f \mapsto 1 \otimes f,$$

is a monomorphism of differential complexes.

The cohomology of the complex $A_\psi(C, V)$ is denoted by $H_\psi(C, V)$ and is called an *entwined cohomology of* C *with values in* V.

As an example we compute the (canonical) entwined cohomology of an A-Galois coextension.

Proposition 3.3 *Let $(A,C)_\psi$ be the canonical entwining structure associated to an algebra-Galois coextension $C(B)_A$ in Example 1.2, and let V be a C-bicomodule. Then*

(1) $H^0_\psi(C,V) = V_B := \{v \in V \mid (\pi \otimes V) \circ {}_V\rho(v) = (V \otimes \pi) \circ \rho_V(v)\}$. Here $\pi : C \to B = C/I$ is the canonical epimorphism.

(2) If $B = k$, then $H^n_\psi(C,V) = 0$, for all $n > 0$.

Proof. The proof is dual to that of Proposition 2.6. The isomorphism in part (1) is

$$H^0_\psi(C,V) \to V_B, \quad f \mapsto \epsilon \circ \rho_C \circ (C \otimes f) \circ {}^V\rho,$$

while the contracting homotopy in part (2) is $h^n : A^n_\psi(C,V) \to A^{n-1}_\psi(C,V)$,

$$h^n(f) = (\hat{\tau} \otimes C^{n-1})(C \otimes \epsilon \otimes C^n)(C \otimes \rho_C \otimes C^n)(\Delta \otimes f)^V\rho,$$

where $\hat{\tau} = (\epsilon \otimes A) \circ \mathrm{cocan}^{-1}$ is the *cotranslation map.* □

Yet another example is the zero-cohomology group of the canonical entwining structure associated to a C-Galois extension.

Example 3.4 *Suppose $(A,C)_\psi$ is the canonical entwining structure associated to a C-Galois extension $A(B)^C$. View C as a C-bicomodule via the coproduct. Then $H^0_\psi(C,C) = {}_B\mathrm{End}^C(A)$ (the space of left B-module right C-comodule endomorphisms of A).*

Proof. In general $H^0_\psi(C,C) = \{\phi \in \mathrm{Hom}(C,A) \mid \forall c \in C,\ \phi(c_{(2)})_\alpha \otimes c_{(1)}{}^\alpha = \phi(c_{(1)}) \otimes c_{(2)}\}$. Now, [2, Theorem 2.4] yields the assertion. □

4 Cup products on $C_\psi(A)$.

The structure of the entwined cohomology of an algebra is particularly rich when the cohomology takes its values in the algebra itself. The situation is very much reminiscent of the Hochschild cohomology of an algebra, as described in [7]. In this section we study the structure of $C_\psi(A) := C_\psi(A,A)$ along the lines of [7].

There are (at least) two ways of defining an associative algebra structure or cup products on $C_\psi(A)$. For any $f \in C^m_\psi(A)$, $g \in C^n_\psi(A)$ define

$$f \cup g = \mu \circ (f \otimes g) \circ (\rho^m_R \otimes A^n) \in C^{m+n}_\psi(A),$$

where ρ^m_R is the right coaction of C on $C \otimes A^m$ described in the introduction. Using the α-notation for the entwining map we can write explicitly,

$$(f \cup g)(c, a^1, \ldots, a^{m+n}) = f(c_{(1)}, a^1_{\alpha_1}, \ldots, a^m_{\alpha_m}) g(c_{(2)}{}^{\alpha_1 \ldots \alpha_m}, a^{m+1}, \ldots, a^{m+n}).$$

We have

Lemma 4.1 *$C_\psi(A)$ is a graded associative algebra with the product $\cup$. Furthermore, d is a degree 1 derivation in this algebra, i.e., for all $f \in C^m_\psi(A)$, $g \in C^n_\psi(A)$,*

$$d(f \cup g) = df \cup g + (-1)^m f \cup dg.$$

Proof. This follows from Proposition 4.5 and Proposition 4.10 below, but it can also be proven by straightforward manipulations with the definition of an entwining structure. In particular, the associativity of $\cup$ follows from the right pentagon in the bow-tie diagram, while to prove the derivation property of d one needs to use the definition of d and both pentagons in the bow-tie diagram. □

Lemma 4.1 implies that the cup product $\cup$ defines the product in the cohomology $H_\psi(A) := H_\psi(A, A)$. This product is also denoted by $\cup$.

The second type of a cup product is defined as follows. For any $f \in C^m_\psi(A)$, $g \in C^n_\psi(A)$, consider

$$f \sqcup g = \mu \circ (A \otimes g) \circ (\psi \otimes A^n) \circ (C \otimes f \otimes A^n) \circ (\Delta \otimes A^{m+n}) \in C_\psi(A)^{m+n}.$$

Explicitly,

$$(f \sqcup g)(c, a^1, \ldots, a^{m+n}) = f(c_{(2)}, a^1, \ldots, a^m)_\alpha g({c_{(1)}}^\alpha, a^{m+1}, \ldots, a^{m+n}).$$

Lemma 4.2 *$C_\psi(A)$ is an associative algebra with the product $\sqcup$. Furthermore, d is a degree 1 derivation in this algebra.*

Proof. This lemma follows from Proposition 4.5 and Proposition 4.10 below too, but it can also be proven by a straightforward application of definitions of d and ψ, which uses both pentagons of the bow-tie diagram. □

Lemma 4.2 implies that the product $\sqcup$ defines the product in $H_\psi(A)$. This product is also denoted by $\sqcup$.

Both cup products are closely related to each other in the cohomology $H_\psi(A)$. This is described in the following

Theorem 4.3 *For all cohomology classes $\xi \in H^m_\psi(A)$, $\eta \in H^n_\psi(A)$,*

$$\xi \cup \eta = (-1)^{mn} \eta \sqcup \xi.$$

Theorem 4.3 is a generalisation of the result of Gerstenhaber for the Hochschild cohomology of an algebra in [7]. Indeed, take the trivial entwining structure $(A, k)_\psi$ (i.e., ψ is a usual flip canonically identified with the identity automorphism of A). In this case $C_\psi(A)$ is simply the Hochschild complex of A and both the cup products $\cup$ and $\sqcup$ become the standard cup product in the Hochschild complex. Theorem 4.3 is then a simple consequence of [7, Section 7, Corollary 1].

The rest of this section is devoted to the proof of Theorem 4.3. We employ a method similar to the one used in [7]. The method used there is based on pre-Lie systems and pre-Lie algebras. Although we are not able to associate a pre-Lie system to the general complex $C_\psi(A)$, still it is possible to construct a system whose properties suffice to prove Theorem 4.3. First we introduce the following generalisation of the notion of a comp algebra introduced in [8][7]

Definition 4.4 *A* (right) weak comp algebra $(V^\bullet, \diamond, \pi)$ *consists of a sequence of vector spaces $V^0, V^1, V^2, \ldots$, an element $\pi \in V^2$ and k-linear operations $\diamond_i : V^m \otimes V^n \to V^{m+n-1}$ for $i \geq 0$ such that for any $f \in V^m$, $g \in V^n$, $h \in V^p$,*

(1) $f \diamond_i g = 0$ if $i > m-1$;

(2) $(f \diamond_i g) \diamond_j h = f \diamond_i (g \diamond_{j-i} h)$ if $i \leq j < n+i$;

(3) if either $g = \pi$ or $h = \pi$,

$$(f \diamond_i g) \diamond_j h = (f \diamond_j h) \diamond_{i+p-1} g \quad \textit{if } j < i;$$

(4) $\pi \diamond_0 \pi = \pi \diamond_1 \pi$.

The weak comp algebra is a (right) comp algebra in the sense of Gerstenhaber and Schack [8] if the condition (3) holds for all g and h. In the case of a comp algebra one can define an associative cup product in $V = \bigoplus_{i=0} V^i$. We shall see that this cup product splits into two cup products for the weak comp algebra. Notice that condition (3) of Definition 4.4 implies also that if either $g = \pi$ or $h = \pi$,

$$(f \diamond_i g) \diamond_j h = (f \diamond_{j-n+1} h) \diamond_i g \quad \text{if } j \geq n+i.$$

Given a weak comp algebra $(V^\bullet, \diamond, \pi)$ define an operation $\diamond : V^m \otimes V^n \to V^{m+n-1}$, for all $f \in V^m$, $g \in V^n$ given by

$$f \diamond g = \sum_{i=0}^{m-1} (-1)^{i(n-1)} f \diamond_i g.$$

Furthermore one can introduce two additional operations $\cup, \sqcup : V^m \otimes V^n \to V^{m+n}$ for all $f \in V^m$, $g \in V^n$ given by

$$f \cup g = (\pi \diamond_0 f) \diamond_m g, \qquad f \sqcup g = (\pi \diamond_1 g) \diamond_0 f. \tag{1}$$

The importance of these operations is revealed in the following

Proposition 4.5 *Given a weak comp algebra $(V^\bullet, \diamond, \pi)$, $V = \bigoplus_{i=0} V^i$ is a (non-unital) graded associative algebra with respect to each of the operations $\cup$ and $\sqcup$ defined in equations (1).*

Proof. We prove this proposition for the operation $\cup$, the proof for $\sqcup$ is analogous. Take any $f \in V^m$, $g \in V^n$ and $h \in V^p$. To prove the associativity of $\cup$ we need to show that $(f \cup g) \cup h = (\pi \diamond_0 ((\pi \diamond_0 f) \diamond_m g)) \diamond_{m+n} h$ equals to $f \cup (g \cup h) = (\pi \diamond_0 f) \diamond_m ((\pi \diamond_0 g) \diamond_n h)$. First using condition (2) in Definition 4.4 one easily finds that $(\pi \diamond_0 f) \diamond_m ((\pi \diamond_0 g) \diamond_n h) = ((\pi \diamond_0 f) \diamond_m (\pi \diamond_0 g)) \diamond_{m+n} h$, so that only the equality $\pi \diamond_0 ((\pi \diamond_0 f) \diamond_m g) = (\pi \diamond_0 f) \diamond_m (\pi \diamond_0 g)$ needs to be shown. We have

$$\begin{aligned} \pi \diamond_0 ((\pi \diamond_0 f) \diamond_m g) &= (\pi \diamond_0 (\pi \diamond_0 f)) \diamond_m g = ((\pi \diamond_0 \pi) \diamond_0 f) \diamond_m g = ((\pi \diamond_1 \pi) \diamond_0 f) \diamond_m g \\ &= ((\pi \diamond_0 f) \diamond_m \pi) \diamond_m g = (\pi \diamond_0 f) \diamond_m (\pi \diamond_0 g), \end{aligned}$$

where we used Definition 4.4(2) to derive the first, second and the fifth equalities, Definition 4.4(4) to derive the third equality and Definition 4.4(3) to obtain the fourth one. □

Conditions in Definition 4.4 allow one to use the same method as in the proof of [7, Theorem 2] to prove the following

Proposition 4.6 *Let $(V^\bullet, \diamond, \pi)$ be a weak comp algebra. Then for all $f \in V^m$, $g \in V^n$ and $h \in V^p$ we have:*

(1) if either f, g or h is equal to π then

$$(f \diamond g) \diamond h - f \diamond (g \diamond h) = \sum (-1)^{i(n-1)+j(p-1)} (f \diamond_i g) \diamond_j h,$$

where the sum is over those i and j with either $0 \le j \le i-1$ or $n+i \le j \le m+n-2$.

(2) $(f \diamond g) \diamond \pi - f \diamond (g \diamond \pi) = (-1)^{n-1} ((f \diamond \pi) \diamond g - f \diamond (\pi \diamond g))$.

Proposition 4.6 allows one to construct a coboundary in a weak comp algebra.

Proposition 4.7 *A weak comp algebra $(V^\bullet, \diamond, \pi)$ is a cochain complex with a coboundary $d : V^m \to V^{m+1}$, $df = (-1)^{m-1}\pi \diamond f - f \diamond \pi$. Furthermore, d is a degree one derivation in both algebras $(V, \cup)$ and $(V, \sqcup)$*

Proof. The first part is a simple consequence of Proposition 4.6 and easily verifiable fact that $\pi \diamond \pi = 0$. The second part can be proven by direct computation. We display it for the cup product $\cup$. Explicitly, for any $f \in V^m$, $g \in V^n$ one needs to show that $d(f \cup g) = df \cup g + (-1)^m f \cup dg$. Using definitions of $\cup$, $\diamond$ and d this amounts to showing that

$$\Gamma_1 = (-1)^{m+n-1}\pi \diamond_0 ((\pi \diamond_0 f) \diamond_m g) + \pi \diamond_1 ((\pi \diamond_0 f) \diamond_m g) - \sum_{j=0}^{m+n-1} (-1)^j ((\pi \diamond_0 f) \diamond_m g) \diamond_j \pi$$

is equal to

$$\begin{aligned}\Gamma_2 \;\; = \;\; & (-1)^{m+n-1}(\pi \diamond_0 f) \diamond_m (\pi \diamond_0 g) + (\pi \diamond_0 (\pi \diamond_1 f)) \diamond_{m+1} g \\ & - \sum_{i=0}^{m-1}(-1)^i(\pi \diamond_0 (f \diamond_i \pi)) \diamond_{m+1} g - (-1)^m \sum_{k=0}^{n-1}(-1)^k(\pi \diamond_0 f) \diamond_m (g \diamond_k \pi) \\ & +(-1)^{m-1}((\pi \diamond_0 (\pi \diamond_0 f)) \diamond_{m+1} g - (\pi \diamond_0 f) \diamond_m (\pi \diamond_1 g)).\end{aligned}$$

The first term in Γ_1 equals the first term in Γ_2 by the same calculation as in the proof of Proposition 4.5. Using a chain of arguments as in the proof of Proposition 4.5 but without the fourth step, one easily shows that the second term in Γ_1 is the same as the second term in Γ_2. Again, a part of the argument in the proof of Proposition 4.5 allows one to transform the first term inside the final brackets in Γ_2 to the form $((\pi \diamond_0 f) \diamond_m \pi) \diamond_{m+1} g$ and then Definition 4.4(2) implies that the last line in Γ_2 vanishes. Now consider the third term in Γ_1. If $j \leq m-1$ then conditions (3) and (2) in Definition 4.4 imply that

$$((\pi \diamond_0 f) \diamond_m g) \diamond_j \pi = ((\pi \diamond_0 f) \diamond_j \pi) \diamond_{m+1} g = (\pi \diamond_0 (f \diamond_j \pi)) \diamond_{m+1} g,$$

so that the part of the sum in the last term in Γ_1 for $j \leq m-1$ is the same as the third term in Γ_2. Similarly, Definition 4.4(2) implies that the remaining part of this sum is the same as the fourth term in Γ_2. This completes the proof that d is a derivation in the algebra $(V, \cup)$. Similar arguments show that d is a derivation in the algebra $(V, \sqcup)$. $\square$

The relationship between three operations $\diamond$, $\cup$ and $\sqcup$ is revealed in the following

Theorem 4.8 *Let $(V^\bullet, \diamond, \pi)$ be a weak comp algebra. Then for all $f \in V^m$, $g \in V^n$:*

$$f \diamond dg - d(f \diamond g) + (-1)^{n-1} df \diamond g = (-1)^{n-1} \left(g \sqcup f - (-1)^{mn} f \cup g \right).$$

Proof. Expand the left hand side of the above equality using definition of d in Proposition 4.7. This produces six terms, four of which cancel because of Proposition 4.6(2). One is left to show that $(\pi \diamond f) \diamond g - \pi \diamond (f \diamond g) = (-1)^{m-1} \left(g \sqcup f - (-1)^{mn} f \cup g \right)$. By Proposition 4.6(1), $(\pi \diamond f) \diamond g - \pi \diamond (f \diamond g) = (-1)^{m-1}((\pi \diamond_1 f) \diamond_0 g - (-1)^{mn} (\pi \diamond_0 f) \diamond_m g)$, therefore the required equality holds once the definition of cup products in equation (1) is taken into account. □

By Proposition 4.7 the cohomology of a weak comp algebra can be equipped with two algebra structures corresponding to products $\cup$ and $\sqcup$. Now Theorem 4.8 implies

Corollary 4.9 *Let $(V^\bullet, \diamond, \pi)$ be a weak comp algebra and let $H(V)$ denote its cohomology with respect to the coboundary operator defined in Proposition 4.7. Then for all cohomology classes $\xi \in H^m(V)$, $\eta \in H^n(V)$,*

$$\xi \cup \eta = (-1)^{mn} \eta \sqcup \xi.$$

Proof. If f and g are cocycles one has $d(f \diamond g) = (-1)^n \left(g \sqcup f - (-1)^{mn} f \cup g \right)$, and the corollary follows. □

Now we are ready to prove Theorem 4.3 by associating a weak comp algebra to the complex $C_\psi(A)$. First, for any $f \in C^m_\psi(A)$, $g \in C^n_\psi(A)$, define $f \diamond_i g \in C^{m+n-1}_\psi(A)$ by

$$f \diamond_i g = \begin{cases} f \circ (C \otimes A^i \otimes g \otimes A^{m-i-1}) \circ (\rho^i_R \otimes A^{m+n-i-1}) & \text{if } 0 \leq i < m \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Explicitly for all $0 \leq i < m$,

$$\begin{aligned} &(f \diamond_i g)(c, a^1, \ldots, a^{m+n-1}) \\ &\qquad = f(c_{(1)}, a^1_{\alpha_1}, \ldots, a^i_{\alpha_i}, g({c_{(2)}}^{\alpha_1 \ldots \alpha_i}, a^{i+1}, \ldots, a^{i+n}), a^{n+i+1}, \ldots, a^{m+n-1}). \end{aligned}$$

Next consider the two-coboundary $\pi \in C^2_\psi(A)$ given by $\pi = \epsilon \otimes \mu$. It is a coboundary, since one easily checks that $\pi = d(\epsilon \otimes A)$.

Proposition 4.10 $(C^\bullet_\psi(A), \diamond, \pi)$ *is a weak comp algebra.*

Proof. Condition (1) in Definition 4.4 is clearly satisfied. Definition 4.4(2) can be proven by a straightforward calculation which uses the right pentagon in the bow-tie diagram. Next, notice that for any $f \in C^m_\psi(A)$

$$f \diamond_i \pi = f \circ (C \otimes A^i \otimes \mu \otimes A^{m-i-1}),$$

take $0 \leq j \leq i-1$ and compute

$$\begin{aligned}
(f \diamond_i g) \diamond_j \pi \;&=\; (f \diamond_i g)(C \otimes A^j \otimes \mu \otimes A^{m+n-j-1}) \\
&= f(C \otimes A^i \otimes g \otimes A^{m-i-1})(\rho^i_R \otimes A^{m+n-i-1})(C \otimes A^j \otimes \mu \otimes A^{m+n-j-1}) \\
&= f(C \otimes A^i \otimes g \otimes A^{m-i-1})(C \otimes A^j \otimes \mu \otimes A^{i-j-1} \otimes C \otimes A^{m+n-i})(\rho^{i+1}_R \otimes A^{m+n-i}) \\
&= f(C \otimes A^j \otimes \mu \otimes A^{m-j-1})(C \otimes A^{i+1} \otimes g \otimes A^{m-i-1})(\rho^{i+1}_R \otimes A^{m+n-i}) \\
&= (f \diamond_j \pi)(C \otimes A^{i+1} \otimes g \otimes A^{m-i-1})(\rho^{i+1}_R \otimes A^{m+n-i}) \\
&= (f \diamond_j \pi) \diamond_{i+1} g,
\end{aligned}$$

where we used Lemma 1.3 to derive the third equality. This proves Definition 4.4(3) with $h = \pi$. The case $g = \pi$ is proven in a similar way. The proof of condition Definition 4.4(4) is again straightforward. □

One can easily verify that the cup products in $C_\psi(A)$ defined at the beginning of this section, are given by equations (1) while the coboundary operator is given by the formula in Proposition 4.7, with operations $\diamond_i$ defined in equation (2). Therefore Theorem 4.3 immediately follows from Corollary 4.9.

Dually, one can associate a weak comp algebra to a C-valued entwined cochain complex of C, $A_\psi(C) = A_\psi(C, C)$. In this case the operations $\diamond_i$ are defined as

$$f \diamond_i g = \begin{cases} (\rho^R_i \otimes C^{m+n-i-1}) \circ (A \otimes C^i \otimes g \otimes C^{m-i-1}) \circ f & \text{if } 0 \leq i < m \\ 0 & \text{otherwise} \end{cases}$$

for all $f \in A^m_\psi(C)$, $g \in A^n_\psi(C)$, while $\pi = 1 \otimes \Delta$.

5 The ψ-equivariant cohomology of A

Guided by the results of the previous section, we construct here a subcomplex $C_{\psi-e}(A)$ of $C_\psi(A)$ whose cohomology has a graded-commutative algebra structure given by the cup product.

For any $n \in \mathbf{Z}_{\geq 0}$ consider a vector subspace of $C^n_\psi(A)$,

$$C^n_{\psi-e}(A) = \{f \in \mathrm{Hom}(C \otimes A^n, A) \mid (f \otimes C) \circ \rho^n_R = \psi \circ (C \otimes f) \circ \rho^n_L\} \subseteq C^n_\psi(A).$$

Lemma 5.1 *$(C^\bullet_{\psi-e}(A), \diamond, \pi)$ is a weak comp algebra with $\diamond_i$ given by equation (2) and $\pi = \epsilon \otimes \mu$.*

Proof. The condition for f to be in $C^n_{\psi-e}(A)$ can be explicitly written for all $c \in C$, $a^1, \ldots, a^n \in A$

$$f(c_{(1)}, a^1_{\alpha_1}, \ldots, a^n_{\alpha_n}) \otimes {c_{(2)}}^{\alpha_1 \cdots \alpha_n} = f(c_{(2)}, a^1, \ldots, a^n)_\alpha \otimes {c_{(1)}}^\alpha. \tag{3}$$

If $f = \pi$ this is precisely the left pentagon in the bow-tie diagram. Therefore $\pi \in C^2_{\psi-e}(A)$. We will show that for all $f \in C^m_{\psi-e}(A)$, $g \in C^n_{\psi-e}(A)$, $f \diamond_i g \in C^{m+n-1}_{\psi-e}(A)$. Take any $c \in C$, $a^1, \ldots, a^{m+n-1} \in A$ and compute

$$\begin{aligned}
f \diamond_i g(c_{(1)}, a^1_{\alpha_1}, \ldots, a^{m+n-1}_{\alpha_{m+n-1}}) \otimes {c_{(2)}}^{\alpha_1 \cdots \alpha_{m+n-1}} \quad &= \\
= f(c_{(1)}, a^1_{\alpha_1\beta_1}, \ldots, a^i_{\alpha_i\beta_i}, g({c_{(2)}}^{\beta_1 \ldots \beta_i}, a^{i+1}_{\alpha_{i+1}}, \ldots, a^{n+i}_{\alpha_{n+i}}), a^{n+i+1}_{\alpha_{n+i+1}}, \ldots, a^{m+n-1}_{\alpha_{m+n-1}})& \\
\otimes {c_{(3)}}^{\alpha_1 \cdots \alpha_{m+n-1}}& \\
= f(c_{(1)}, a^1_{\alpha_1}, \ldots, a^i_{\alpha_i}, g({{c_{(2)}}^{\alpha_1 \ldots \alpha_i}}_{(1)}, a^{i+1}_{\alpha_{i+1}}, \ldots, a^{n+i}_{\alpha_{n+i}}), a^{n+i+1}_{\alpha_{n+i+1}}, \ldots, a^{m+n-1}_{\alpha_{m+n-1}})& \\
\otimes {{{c_{(2)}}^{\alpha_1 \ldots \alpha_i}}_{(2)}}^{\alpha_{i+1} \ldots \alpha_{m+n-1}}& \\
= f(c_{(1)}, a^1_{\alpha_1}, \ldots, a^i_{\alpha_i}, g({{c_{(2)}}^{\alpha_1 \ldots \alpha_i}}_{(2)}, a^{i+1}, \ldots, a^{n+i})_\beta, a^{n+i+1}_{\alpha_{n+i+1}}, \ldots, a^{m+n-1}_{\alpha_{m+n-1}})& \\
\otimes {{{c_{(2)}}^{\alpha_1 \ldots \alpha_i}}_{(1)}}^{\beta\alpha_{n+i+1} \ldots \alpha_{m+n-1}}& \\
= f(c_{(1)}, a^1_{\alpha_1\beta_1}, \ldots, a^i_{\alpha_i\beta_i}, g({c_{(3)}}^{\alpha_1 \ldots \alpha_i}, a^{i+1}, \ldots, a^{n+i})_\beta, a^{n+i+1}_{\alpha_{n+i+1}}, \ldots, a^{m+n-1}_{\alpha_{m+n-1}})& \\
\otimes {c_{(2)}}^{\beta_1 \ldots \beta_i\beta\alpha_{n+i+1} \ldots \alpha_{m+n-1}}& \\
= f(c_{(2)}, a^1_{\alpha_1}, \ldots, a^i_{\alpha_i}, g({c_{(3)}}^{\alpha_1 \ldots \alpha_i}, a^{i+1}, \ldots, a^{n+i}), a^{n+i+1}, \ldots, a^{m+n-1})_\alpha \otimes {c_{(1)}}^\alpha& \\
= f \diamond_i g(c_{(1)}, a^1, \ldots, a^{m+n-1})_\alpha \otimes {c_{(2)}}^\alpha,&
\end{aligned}$$

where the right pentagon in the bow-tie diagram has been used in derivation of the second and fourth equalities, and equation (3) for f and g in derivation of the third and fifth equalities. □

Therefore $(C^\bullet_{\psi-e}(A), \diamond, \pi)$ is a weak comp subalgebra of $(C^\bullet_\psi(A), \diamond, \pi)$. Consequently, $C_{\psi-e}(A) = (C^\bullet_{\psi-e}(A), d)$ is a cochain subcomplex of $C_\psi(A)$. The corresponding cohomology is denoted by $H_{\psi-e}(A)$ and called *ψ-equivariant cohomology of A*. As for any

weak comp algebra one can define the cup products in $C_{\psi-e}(A)$ which will descend to the cohomology $H_{\psi-e}(A)$. Notice, however, that $\cup$ coincides with $\sqcup$ in $C_{\psi-e}(A)$, and, consequently in $H_{\psi-e}(A)$. Therefore, as a simple consequence of Theorem 4.3 and Theorem 4.8 one obtains the following

Theorem 5.2 *For all $f \in C^m_{\psi-e}(A)$, $g \in C^n_{\psi-e}(A)$:*

$$f \diamond dg - d(f \diamond g) + (-1)^{n-1} df \diamond g = (-1)^{n-1} \left(g \cup f - (-1)^{mn} f \cup g\right).$$

Consequently the algebra $(H_{\psi-e}(A), \cup)$ is graded-commutative.

As an example of a ψ-equivariant complex we consider such a complex associated to the canonical entwining structure.

Take any A-bimodule M with the left and right actions ${}_M\rho$ and ρ_M respectively, and consider two operations

$$\mathrm{Hom}(C \otimes A^n, A) \otimes \mathrm{Hom}(C, M) \to \mathrm{Hom}(C \otimes A^n, M), \quad f \otimes \phi \mapsto f \cup \phi := {}_M\rho \circ (f \otimes \phi) \circ \rho^n_R,$$

$$\mathrm{Hom}(C, M) \otimes \mathrm{Hom}(C \otimes A^n, A) \to \mathrm{Hom}(C \otimes A^n, M), \quad \phi \otimes f \mapsto \phi * f := \rho_M \circ (\phi \otimes f) \circ \rho^n_L.$$

Example 5.3 *Let $(A, C)_\psi$ be the canonical entwining structure associated to a C-Galois extension $A(B)^C$, and let $\tau = \mathrm{can}^{-1} \circ (1 \otimes C)$ be the translation map. Then $f \in C^n_{\psi-e}(A)$ if and only if $f \cup \tau = \tau * f$.*

Proof. Apply can^{-1} to (3) and then use the definitions of the canonical entwining structure and the translation map to obtain

$$\begin{aligned} (f \cup \tau)(c, a^1, \ldots, a^n) &= f(c_{(2)}, a^1, \ldots, a^n)_\alpha \tau({c_{(1)}}^\alpha) \\ &= {c_{(1)}}^{(1)}({c_{(1)}}^{(2)} f(c_{(2)}, a^1, \ldots, a^n))_{(0)} \tau(({c_{(1)}}^{(2)} f(c_{(2)}, a^1, \ldots, a^n))_{(1)}) \\ &= \tau(c_{(1)}) f(c_{(2)}, a^1, \ldots, a^n) = (\tau * f)(c, a^1, \ldots, a^n), \end{aligned}$$

where $\tau(c) = c^{(1)} \otimes c^{(2)}$. $\square$

6 Deformation of entwining structures

In this section we associate a double-complex to any entwining structure. The total cohomology of a particular subcomplex of this complex gives the cohomological interpretation of deformation theory of entwining structures.

Given an entwining structure $(A, C)_\psi$, let $C(A, C, \psi) = (C^{\bullet,\bullet}(A, C, \psi), d, \bar{d})$ be a double complex obtained by applying the Hom-functor in the category of A-bimodules, C-bicomodules to the ψ-twisted bar and cobar resolutions, i.e.,

$$C(A, C, \psi) = {}_A^C\mathrm{Hom}_A^C(\mathrm{Bar}^\psi(A), \mathrm{Cob}_\psi(C)).$$

Here $A \otimes C \otimes A^{n+1}$ is viewed as a C-bicomodule via $(\sigma \otimes C \otimes A^{n+1}) \circ (A \otimes \rho_L^n \otimes A)$ and $(A \otimes C \otimes A^n \otimes \sigma) \circ (A \otimes \rho_R^n \otimes A)$, where σ is a flip. Dually $C \otimes A \otimes C^{n+1}$ is viewed as an A-bimodule via $(C \otimes \rho_n^L \otimes C) \circ (\sigma \otimes A \otimes C^{n+1})$, and $(C \otimes \rho_n^R \otimes C) \circ (C \otimes A \otimes C^{n+1} \otimes \sigma)$. Clearly bimodule structures commute with the bicomodule structures. Explicitly we have

$$C^{m,n}(A, C, \psi) = C_\psi^m(A, A \otimes C^n) = A_\psi^n(C, C \otimes A^m) = \mathrm{Hom}(C \otimes A^m, A \otimes C^n),$$

$$d : C^{m,n}(A, C, \psi) \to C^{m+1,n}(A, C, \psi), \qquad \bar{d} : C^{m,n}(A, C, \psi) \to C^{m+1,n}(A, C, \psi),$$

where d is a coboundary operator described in Section 2, corresponding to the entwined cohomology of A with values in the A-module $(A \otimes C^n, \rho_n^L, \rho_n^R)$, while $\bar{d}$ is the coboundary operator of Section 3, corresponding to the entwined cohomology of C with values in the C-bicomodule $(C \otimes A^m, \rho_L^m, \rho_R^m)$. It follows directly from the construction that $d\bar{d} = \bar{d}d$, so that $C(A, C, \psi)$ is a double cochain complex as claimed, with the total coboundary operator $D^{m,n} : C^{m,n}(A, C, \psi) \to C^{m+1,n}(A, C, \psi) \oplus C^{m,n+1}(A, C, \psi)$, $D^{m,n} = d + (-1)^m \bar{d}$.

From the point of view of deformation of entwining structures the following modification of the double complex $C(A, C, \psi)$ is of substantial importance. Let $C_H^{m,0}(A, C, \psi) = \mathrm{Hom}(A^m, A)$, $C_H^{0,n}(A, C, \psi) = \mathrm{Hom}(C, C^n)$, $C_H^{m,n}(A, C, \psi) = \mathrm{Hom}(C^{\otimes} A^m, A \otimes C^n)$, for $m, n \geq 1$. In other words, the complex $C_H(A, C, \psi)$ is obtained from $C(A, C, \psi)$ by replacing the first line and first column with the Hochschild complex of A and the Cartier complex of C viewed in $C(A, C, \psi)$ via the monomorphisms j in Lemma 2.2 and $\bar{j}$ in Lemma 3.2. Thus, explicitly, the complex $C_H(A, C, \psi)$ is

$$\begin{tikzcd}
 & \mathrm{Hom}(A, A) \arrow[r, "d"] \arrow[d, "\bar{d} \circ j"'] & \mathrm{Hom}(A^2, A) \arrow[r, "d"] \arrow[d, "\bar{d} \circ j"'] & \mathrm{Hom}(A^3, A) \arrow[r, "d"] \arrow[d, "\bar{d} \circ j"'] & {} \\
\mathrm{Hom}(C, C) \arrow[r, "d \circ \bar{j}"] \arrow[d, "\bar{d}"'] & \mathrm{Hom}(C \otimes A, A \otimes C) \arrow[r, "d"] \arrow[d, "\bar{d}"'] & \mathrm{Hom}(C \otimes A^2, A \otimes C) \arrow[r, "d"] \arrow[d, "\bar{d}"'] & \mathrm{Hom}(C \otimes A^3, A \otimes C) \arrow[r, "d"] \arrow[d, "\bar{d}"'] & {} \\
\mathrm{Hom}(C, C^2) \arrow[r, "d \circ \bar{j}"] \arrow[d, "\bar{d}"'] & \mathrm{Hom}(C \otimes A, A \otimes C^2) \arrow[r, "d"] \arrow[d, "\bar{d}"'] & \mathrm{Hom}(C \otimes A^2, A \otimes C^2) \arrow[r, "d"] \arrow[d, "\bar{d}"'] & \mathrm{Hom}(C \otimes A^3, A \otimes C^2) \arrow[r, "d"] \arrow[d, "\bar{d}"'] & {} \\
{} & {} & {} & {} &
\end{tikzcd}$$

This double complex combines into a complex $C_H((A,C)_\psi) = (C_H^\bullet((A,C)_\psi), D)$,

$$C_H^n((A,C)_\psi) = \mathrm{Hom}(A^n, A) \oplus \bigoplus_{k=1}^{n-1} \mathrm{Hom}(C \otimes A^{n-k}, A \otimes C^k) \oplus \mathrm{Hom}(C, C^n).$$

The cohomology of the complex $C_H((A,C)_\psi)$ is denoted by $H_H((A,C)_\psi)$.

The cohomology $H_H((A,C)_\psi)$ plays an important role in the deformation theory of entwining structures. The latter can be developed along the same lines as the deformation theory of algebra factorisations in [1], following the Gerstenhaber deformation programme [6]. We sketch here the main results.

Let $(A,C)_\psi$ be an entwining structure. A *formal deformation* of $(A,C)_\psi$ is an entwining structure $(A_t, C_t)_{\psi_t}$, over the ring $k[[t]]$, where A_t, C_t are algebra and coalgebra deformations of A and C respectively, and $\psi_t = \psi + \sum_{i=1} t^i \psi^{(i)}$, $\psi^{(i)} : C \otimes A \to A \otimes C$. In other words the deformation $(A_t, C_t)_{\psi_t}$ is characterised by three maps expandable in the power series in t,

$$\mu_t = \mu + \sum_{i=1} t^i \mu^{(i)}, \quad \Delta_t = \Delta + \sum_{i=1} t^i \Delta^{(i)}, \quad \psi_t = \psi + \sum_{i=1} t^i \psi^{(i)}. \tag{4}$$

Two deformations $(A_t, C_t)_{\psi_t}$ and $(A_t, C_t)_{\tilde{\psi}_t}$ are *equivalent* to each other if there exist algebra isomorphism $\alpha_t : A_t \to \tilde{A}_t$ and a coalgebra isomorphism $\gamma_t : C_t \to \tilde{C}_t$ of the form $\alpha_t = A + \sum_{i=1} t^i \alpha^{(i)}$, $\gamma_t = C + \sum_{i=1} t^i \gamma^{(i)}$, and such that $\tilde{\psi}_t \circ (\gamma_t \otimes \alpha_t) = (\alpha_t \otimes \gamma_t) \circ \psi_t$. A deformation $(A_t, C_t)_{\psi_t}$ is called a *trivial deformation* if it is equivalent to an entwining structure over $k[[t]]$ in which all the maps $\mu^{(i)}$, $\Delta^{(i)}$, $\psi^{(i)}$ in (4) vanish. An *infinitesimal deformation* of $(A,C)_\psi$ is a deformation of $(A,C)_\psi$ modulo t^2.

Using similar method to [1, Theorem 3.1] one proves the following

Theorem 6.1 *There is a one-to-one correspondence between the equivalence classes of infinitesimal deformations of* $(A,C)_\psi$ *and* $H_H^2((A,C)_\psi)$.

Theorem 6.1 is a standard result in various deformation programmes. The appearance of the total cohomology of a double complex makes the deformation theory of entwining structures similar to the deformation theory of bialgebras [9].

Furthermore, one can look at the obstructions for extending a deformation of an entwining structure modulo t^n to a deformation modulo t^{n+1}. Not surprisingly one finds that such obstructions are classified by the third cohomology group $H_H^3((A,C)_\psi)$. We leave the details of the analysis of obstructions to the reader, and only mention that the details of a semi-dual case can be found in [1].

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